Coupled Electrical Oscillators

Physics 3600 - Advanced Physics Lab - Summer 2021
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I. INTRODUCTION

The objectives of this experiment are:

(1) explore the properties of a single LRC electrical oscillator circuit, including damping;

(2) study what happens when two oscillators are coupled and allowed to exchange energy.

More information on coupled oscillators can be found in the Appendix.

II. APPARATUS

small box with circuit, switches, connectors, two yellow (or large) inductor coils
capacitors - fixed standard and adjustable
storage scope with interfaced computer
curve fitting software (EastPlot , MatLab, Python)

III. PROCEDURE

A. Measure Inductance

Install the standard capacitor and one inductor in the A-circuit. Connect the +5 VDC source to the box. Connect the A-output to Chnl-1 of the scope. Connect the Trigger output to the scope trigger. Look for the distorted square wave output having oscillations at the edges of the pulse. Find the decaying oscillation of the A-circuit that occurs when the square wave pulse turns **off**. Note that the square wave pulse turning on or off is similar to a hammer striking a bell to start it ringing (oscillating).

1. Measure the capacitance of the standard capacitor using the Extech Digital MultiMeter (DMM).
2. View the oscillation where the excitation pulse goes to V=0, which has the largest number of cycles.
3. Measure only **one** oscillation period of each inductor in the A-circuit.
   a. Compute the two inductances and estimate their uncertainties.
   b. Compare the values of the inductors.

Using the instructions below, curve fit the oscillations to precisely measure the frequency and damping.

4. Store the *decaying* waveform of one inductor in the A-circuit.
   First, transform the waveform to make the average amplitude equal to zero (σ in EasyPlot).
   Curve-fit one or two periods to y=a*sin(bx+c) to obtain values for the frequency b and phase c.
   Then, knowing initial values for b and c, fit the oscillations to y=a*sin(bx+c)exp(-dx/2).
   a. Plot the V(t) data along with the curve fit.
   b. What is the value of the damping coefficient γ (=d)?
   c. What is the value of the time constant of the damping, $\tau = 2/d$?
   d. Compute precise values for L, R, Q, and uncertainties for the A-circuit? (see appendix)
   e. Compare the inductance uncertainty from the curve fit to that from the single period measurement.

You must show the TA your plots at the end of each section before proceeding to the next section.

B. Compare Resistances

a. Compare the R computed from γ to the R of the coil measured with an ohmmeter. Discuss.
b. According to theory, by what percentage does damping affect the frequency?

C. Independent Oscillators

Add the inductor and adjustable capacitor to the B-circuit and connect the output to Chnl-2 of the scope. With the coupling capacitor switched ***off*** (No) and "normal" excitation, adjust the B-capacitor until both waveforms have the same period.

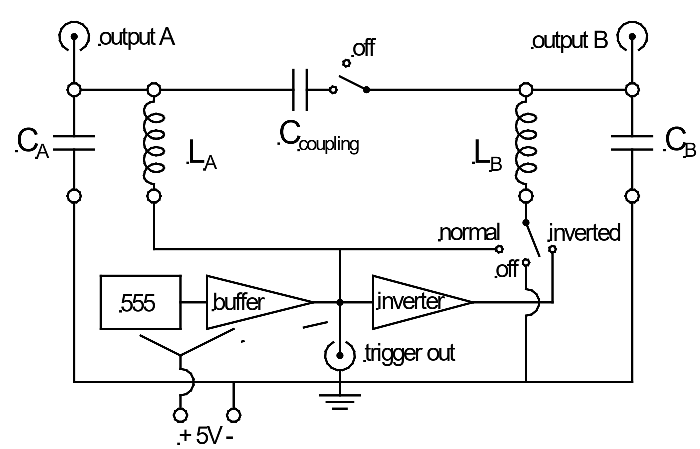
a. Measure the adjustable B-capacitor with the DMM and compare to the standard capacitor.

D. Unilateral Excitation of Coupled Oscillators

With the coupling capacitor ***on*** (Yes), switch off the B-circuit excitation (middle position of switch).

Again, use the oscillations that occur when the exciting pulse goes to zero volts.
Note that both the A and B waveforms periodically grow and decay in time.
Now, finely adjust the adjustable B-capacitor to achieve the best minima of one waveform.

1. Store waveforms of the A- and B-circuits.
2. Plot waveforms for the A- and B-circuits and their sum on one graph (shifted so they don't overlap).
   a. Relate this electrical exchange to that in the mass/spring system in the lab room. Lower the mass, let go of it and describe its motion.
3. Plot the A and B data after removing the decay by dividing by the decaying exponential exp($-\gamma t/2$).
   a. What is the "beat" frequency from the beat period?
   b. How would you curve fit the beats?
   c. Compute the value of the coupling capacitor $C_{coupling}$ from the beat frequency.
   d. Measure $C_{coupling}$ directly (without removing it from the box) and explain how you measured it. (Figure out how to separate it from the circuit.)
4. Discuss what happens to the individual waveforms when the two uncoupled oscillators have slightly different frequencies, but are still coupled?
5. Discuss any differences in the waveforms for the rising and falling edges of the excitation pulse.



IV. APPENDIX: SIMPLE NOTES ON DRIVEN DAMPED HARMONIC OSCILLATOR

This page derives formulas for mechanical and electrical harmonic oscillators that have damping.
The next page derives formulas for a harmonic oscillator (HO) that is driven externally.

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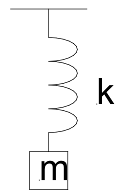
Mechanical HO

The damped *force* equation for the *position* $x(t)$ is



$$m\frac{d^2x}{dt^2}+\beta\frac{dx}{dt}+kx=0.$$

Here, m is the mass, β the velocity-dependent damping constant, and k the spring constant. Using the following substitutions

$\gamma = \beta/m$ and $\omega_o{}^2 = k/m$.

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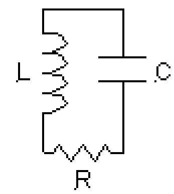
LRC Electrical HO

The *voltage* equation for capacitor *charge* $q(t)$ is



$$L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=0.$$

Here L is the inductance, R is the resistance, C is the capacitance, and i=dq/dt is the current. Using the following substitutions

$\gamma = R/L$ and $\omega_o{}^2 = 1/LC$.

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The above equations can be rewritten generally as $\boxed{\dfrac{d^2x}{dt^2}+\gamma\dfrac{dx}{dt}+\omega_0^2x=0.}$

Solutions are found by substituting x(t)=Re{Ae$^{a\,t}$} into the differential equation, then

$$[\alpha^2+\alpha\,\gamma+\omega_o{}^2]\;x(t)=0,\text{ or }[\alpha 2+\alpha\,\gamma+\omega_o{}^2]=0,\text{ and }\alpha=½\,[-\gamma\pm(\gamma^2-4\,\omega_o{}^2)^{1/2}].$$

The solution for the underdamped case, $\gamma \ll \omega_o$, is a sinusoid with a decaying amplitude given by $\boxed{x(t)=x_0e^{-\frac{\gamma}{2}t}\cos(\omega_0t).}$

The damping constant γ has the same units of frequency as the oscillating frequency ω_o. The angular frequency ω has units of radians/sec, or simply s^{-1}. The frequency f has units of Hertz (cycles/sec), where $\omega=2\pi f$. The period of oscillation is $\boldsymbol{T = 1/f = 2\pi/\omega}$.

Note that the damping reduces the frequency from ω_o to $\omega' = (\omega_o{}^2 - \gamma 2/4)^{1/2} = \omega_o\,(1-\gamma^2/4\,\omega_o{}^2)^{1/2}$.

The "quality factor" or Q-factor is a dimensionless quantity given by the ratio of frequency to damping,

$$\boxed{Q\equiv\frac{\omega_0}{\gamma}.}$$

4

V. Appendix: From Alverson Lab Manuel

II.B Coupled Oscillators

For the sake of mathematical simplicity, the resistance R_p in the oscillators will now be ignored. Although the resistance has a very noticeable effect on the amplitude of oscillation, it has only a small effect on the frequency. In this part of the experiment, only the periods of the various observed motions will be calculated, and therefore the resistance will be taken as zero. Two identical LC oscillators are capacitively coupled together as shown in Figure 4.4.

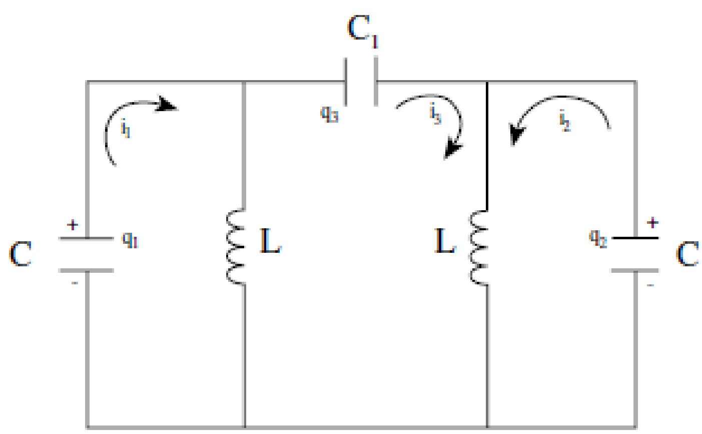


Figure 4.4. Coupled LC Oscillators

The equations describing this circuit are:

$$\frac{q_1}{C} = L\frac{d}{dt}(i_1 - i_3) \tag{4.5}$$

$$\frac{q_2}{C} = L\frac{d}{dt}(i_2 + i_3) \tag{4.6}$$

$$\frac{q_1}{C} = \frac{q_2}{C} - \frac{q_3}{C_1}, \qquad \text{so} \qquad i_3 = \frac{C_1}{C}(i_2 - i_1) \tag{4.7}$$

using the equations in (7) above to eliminate q_3 and i_3 from Eqs. (5) and (6), two coupled equations describing i_1 and i_2 are found:

$$\frac{q_1}{C} = L\frac{d}{dt}\left[i_1\left(1 + \frac{C_1}{C}\right) - \frac{C_1}{C}i_2\right] \tag{4.8}$$

5

$$\frac{\quad}{C} = L\frac{d}{dt}\left[i_2\left(1+\frac{C_1}{C}\right) - \frac{C_1}{C}i_1\right] \tag{4.9}$$

Note the each equation is a function of two variables, namely i_1 and i_2. Taking the sum and difference of Eqs. (8) and (9), two new equations are found:

$$(q_1+q_2)\frac{1}{C} = L\frac{d}{dt}(i_1+i_2) \tag{4.10}$$

$$(q_1-q_2)\frac{1}{C} = \frac{L}{C}(C+2C_1)\frac{d}{dt}(i_1-i_2) \tag{4.11}$$

These equations have the useful property that they can be written as a function of a single variable with the substitutions:

$$q_+ = q_1 + q_2 \qquad q_- = q_1 - q_2$$

Making these substitutions lead to the pair of equations:

$$\frac{d^2}{dt^2}q_+ + \frac{1}{LC}q_+ = 0 \tag{4.12}$$

$$q_+ = q_+^0 \sin\omega_+ t \tag{4.13}$$

$$\frac{d^2}{dt^2}q_- + \frac{1}{L(C+2C_1)}q_- = 0 \tag{4.14}$$

$$q_- = q_-^0 \sin\omega_- t \tag{4.15}$$

$$\omega_- = \sqrt{\frac{1}{L(C+2C_1)}} \tag{4.16}$$

Thus the variables q_+ and q_- each behave as a simple harmonic oscillator with frequencies of oscillation:

$$\frac{1}{2\pi\sqrt{LC}} \quad \text{and} \frac{1}{2\pi\sqrt{L(C+2C_1)}}$$

The variables q_+ and q_- are called the normal coordinates for the system. Eqs. (10) and (11) each describe situations where the coupled oscillators act as a single harmonic oscillator. These situations are called the "normal modes" of the system. In general, the system motion can be more complicated than one of the normal modes. These modes represent rather simple, but special cases of behavior, since the system can always be described as a linear combination of the two modes, *i.e.*, $q = Aq_+ + Bq_-$. The constants A and B are determined by the initial conditions, i.e. the values of q_1, i_1, q_2, and i_2 at $t = 0$. In this experiment three cases will be observed: For all three cases the initial currents will be zero, but the initial voltages, corresponding to the initial values of q_1 and q_2 will be:

$q_1 = q_2$ at $t = 0$. This corresponds to the normal mode q_+ and is called the symmetric mode since each oscillator is oscillating in phase at the same frequency.

$q_1 = -q_2$ at $t = 0$. This is the normal mode q_- and is called the antisymmetric mode since each oscillator is oscillating at 180° out of phase.

$q_1 = q_0, q_2 = 0$ at $t = 0$. In this case, one of the oscillators is given an initial amplitude and the other is not. Here the subsequent behavior of the system is not described by either of the normal modes alone. The behavior is more complex and is described by a sum of equal amplitudes of each mode:

$$q = q_0(\sin\omega_+ t + \sin\omega_- t)$$

Since the two normal mode frequencies are not equal, the resulting sum is not a single harmonic function. For *weak* coupling, ($C_1 < C$), $\omega_+ \approx \omega_-$ and the two normal modes are close in frequency. When two functions of nearly equal frequency and identical amplitudes are added, the resulting behavior has a *beat* phenomena. This can be easily seen by applying trigonometric identities to the previous expression for q:

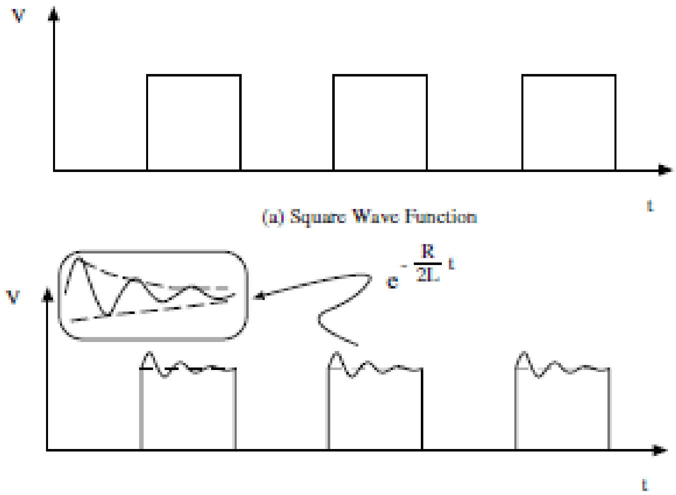
$$q = 2q_0 \sin\left[\left(\frac{\omega_+ + \omega_-}{2}\right)t\right]\cos\left[\left(\frac{\omega_+ - \omega_-}{2}\right)t\right]$$

If $\omega_+ \approx \omega_- \approx \omega_0$, the behavior can be thought of as one sine function of frequency nearly equal to ω_0, but whose amplitude slowly fluctuates between 0 and $2q_0$ at a frequency of $(\omega_+ - \omega_-)/2$. The beat frequency $(\omega_+ - \omega_-)$ is double the naive frequency $(\omega_+ - \omega_-)/2$ as it corresponds to the frequency of successive maximum values. The behavior can be thought of as a pair of oscillators each oscillating at the average of the normal mode frequencies and the energy of oscillation slowly going back and forth between them with the beat frequency.

III. Apparatus

Electrical energy is supplied to the oscillator by means of a square wave which is produced by a timer chip (#555) which is powered by a 5 Volt DC supply. Each transition of the square wave gives an initial amplitude to the oscillator. The period of the square wave is chosen so that there is sufficient time between transitions (Figure 4.5) for the induced oscillations to die down before the next transition occurs. The oscillations are observed by means of an oscilloscope connected across the capacitors. The apparatus for this experiment consists of a 5 volt power supply, a circuit for supplying the appropriate waveform to the oscillators, a capacitance box, two large wirewound inductors, a dual trace oscilloscope, and an oscilloscope camera. The circuit for supplying the appropriate waveforms, one of the oscillator capacitors, and the coupling capacitor are contained in a small grey box. The circuit diagram to this box and its connections are shown in Figure 4.6.

A capacitance box is used to supply the capacitor for the second oscillator and instructions are given below for adjusting its value so that the two LC oscillators are identical.



(a) Square Wave Function

(b) Transient Response to a Square Wave Function

Figure 4.5.

There are several switches on the grey metal box. Switch 1 turns on and off the coupling between the two oscillators. Switch 2 determines whether or not oscillator B will be given an initial amplitude which is in phase (*normal*) or out of phase (*inverted*) with oscillator A. Switch 2 may also be set to *off*, so that no initial amplitude is given to oscillator B.